Review for Midterm I[1]

Assigned: February 19, 2021

Multivariable Calculus MATH 53
with Professor Stankova

Contents

1 Definitions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, and to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition. What is/are:

1. a *parametric* curve? paramteric equations? a *parameter*?
   How do they differ from their *Cartesian* counterparts?
2. a *cycloid*? Can it be parametrized? How?
3. an *ellipse*, a *hyperbola*, a *parabola*? How can each be parametrized?
4. *polar* coordinates? Relation to Cartesian coordinates? How do they differ?
5. a *polar* curve? Can *all* curves be represented as polar curves? In this context, what are the *Archimedian* spiral, the 4-*leaf rose*, the *figure*-8 *lemniscate*, a *cardioid*, an *asteroid*?
6. a *tangent line* to a parametric curve? to a polar curve? to a Cartesian curve?
   Do these tangents lines differ or are they the same line?
7. the *concavity* of a parametric curve?
8. the *arc length* of a parametric curve? of a polar curve?
9. a 3-*dimensional* coordinate system? What is $\mathbb{R}^2$? $\mathbb{R}^3$?
10. a vector $\vec{v}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ from a geometric and from an algebraic point of view?
11. the *sum* and the *difference* of two vectors $\vec{v}$ and $\vec{w}$: geometrically and algebraically?
    How about the *scalar product* of a vector $\vec{v}$ with a scalar c?
12. the *basic properties* of addition, subtraction and scalar multiplication of vectors?
    How do these properties resemble properties of the corresponding operations on *real numbers*?
13. the *length* of a vector $\vec{v}$? How do we calculate it?
14. a *unit* vector? Can we re-scale all vectors to make them unit? How?

15. the *standard* unit vectors in $\mathbb{R}^2$? in $\mathbb{R}^3$? Why are they so important?
16. a *linear combination* of vectors? How many ways can we express a vector as a linear combination of the standard unit vectors? Why?
17. a *median* and the *centroid* of a triangle? How are they related to our study of vectors?
18. the *dot product* of two vectors: algebraically and geometrically? How do we calculate it?
19. the *cross product* of two vectors: algebraically and geometrically? How do we calculate it?
20. the *basic properties* of dot and cross products, in relation to each other and in relation to the other operations on vectors?
21. *orthogonal* vectors? How to determine if two vectors are orthogonal, using vector operations?
22. *parallel* vectors? How do we determine if two vectors are pointing in the same direction, using operations on vectors? What are *colinear* vectors?
23. the *angle* between two vectors? How do we calculate the angle using the two vectors?
24. *orthogonal vector* and *scalar projections* of vectors? How do we calculate them?
25. *direction angles* and *direction cosines* of a vector?
    What is the main relationship between the three directional cosines of a vector?
26. a 2×2 matrix? a 3×3 matrix? the *determinants* of such matrices?
27. *co-planar* vectors? When are three vectors co-planar?
28. the *triple scalar product* of vectors? What is it useful for?
29. the *right-hand rule*? What is it used for?
30. the *point-slope* formula? What kind of a geometric object does it describe?
31. *parametric*, *vector*, and *Cartesian* (*symmetric*) equations for a line in space?
32. the *direction numbers of* a line in space? Are they unique? Why do we need 3 such numbers?
33. a *normal* vector to a plane? *vector*, *scalar*, and *linear* equations for a plane in space?
34. the *angle* between two planes? the *distance* between two planes?
35. a *cylinder*? What are its *base curve*, its *traces*, and its *ruling*?
    A cylinder can be thought of as the *disjoint union* of what objects? In how many ways?
36. a *quadratic equation* in three variables?
    How do we transform it into the standard equations of *quadric surfaces*?
37. the equations for *quadric cylindrical and non-cylindrical surfaces*? How many are they? How do we recognize each? What are their all possible *traces*? Why name them the way we do?
38. a *scalar* vs. a *vector* function? How do the latter relate to parametric curves?
39. the *limit*, *continuity*, *derivative*, and *integral* of a vector function?
    Why do we say that they are defined *component-wise*?
40. the *tangent* vs. a *secant* slope at a point P on the graph of a function $y = f(x)$?
    What is their relation to the derivative $f'(x)$ at P?
41. a *helix*? the *twisted cubic*? On which famous surfaces does each lie?
42. the *tangent vector* and *tangent line* for a vector function? the *unit* tangent vector?
43. the *arc-length* of a parametric curve? the arc-length *function*? How does it relate to velocity and speed? What does it mean to *re-parametrize* wrt arc-length? Why is this parametrization called "universal"? In what ways is it *not* unique?
44. an *intrinsic* feature of a curve? an *extrinsic* feature of a curve? a feature that is *independent* or not of parametrization? Can you list all features of curves we have studied and split them into intrinsic and extrinsic ones?
45. a *smooth* curve? What can we define on a smooth curve that cannot be well-defined on a non-smooth curve? Are the circle, any helix, and the twisted cubic smooth? How about any of the three *projections* of the twisted cubic onto the coordinate planes?
46. the *curvature* of a smooth curve? How does it depend on the parametrization of the curve? How does it relate to the derivative of any unit tangent vector? to the tangent vector wrt to

arc-length parametrization? to the derivative and acceleration vectors for any parametrization of the curve?

47. the curvature of a circle? of a helix? of a plane curve? What are the extreme curvatures along the twisted cubic or the regular cubic $y = x^3$?
48. the *normal* and *binormal* vectors of a vector function? How do they relate to the (unit) tangent vector? Which of them is independent of the parametrization?
49. the *normal*, *osculating*, and *rectifying* planes of a vector function? How do we picture them in relation to the motion of a particle along the corresponding path? Which famous vector is normal to each plane? What famous vectors does each plane contain?
50. the *tangential and normal components* of acceleration? On what does each component depend? Which component is necessarily non-negative and why? Which component can be negative and when does this happen? In which plane does the acceleration vector always lie? Is acceleration independent of parametrization?

2 Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

1. **Conversions back-and-forth between Cartesian and polar coordinates.**
2. **Formulas for the following features of parametric curves and by polar curves:**
   - slopes of tangents to such curves;
   - second derivatives;
   - areas described by these curves, and areas between two such curves;
   - arc lengths of such curves;
   - surface area of solid of revolution given by such a curve.
3. **Properties of the following operations on vectors** (separately and in combinations):
   - addition, subtraction, scalar multiplication; taking linear combinations;
   - taking the magnitude of a vector;
   - dot product; cross product.
4. **Specials vector relations in a triangle**: formulas for the medians and for the sum of vectors from the centroid to each of the three vertices.
5. **Law of Cosines in a Triangle.**
6. **Formula for the dot product:** $\vec{v} \circ \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos \alpha$.
   - Corollary on how to calculate the angle between two vectors.
   - Conditions for vectors to be orthogonal, pointing in the same or opposite directions, making an acute or an obtuse angle.
7. **Formulas for vector and scalar orthogonal projections** of $\vec{v}$ onto $\vec{w}$.
8. **Formulas for direction angles and direction cosines**:
   - Expressing a vector using its direction cosines.
   - Sum of squares of the direction cosines.
9. **The Triangle Inequality:** when is equality obtained?
10. **Formulas for determinants of 2×2 and 3×3 matrices.**
    - Connection to cross products of vectors.
    - Basic properties of determinants that are relevant to cross products.

11. **Formula for the length of the cross product:** $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \sin\alpha$.
    - Corollary on how to calculate the angle between two vectors.
    - Conditions for vectors to be orthogonal, parallel, or making an acute or an obtuse angle.
12. **Formulas for:**
    - the areas of a parallelogram and a triangle; • for the volume of a parallelepiped;
13. **Condition for three vectors to be co-planar:** iff $\vec{v} \circ (\vec{w} \times \vec{u}) = 0$.
14. **Equations for lines:**
    - in the plane: point-slope formula; Cartesian equation;
    - in space: parametric, vector and Cartesian (symmetric) equations.
15. **Equations for planes:** vector, scalar, and linear equations.
    - the angle between planes;
    - distances from a point to a plane, and from a line to a plane, and between planes.
16. **Standard equations for quadric surfaces:**
    - quadric cylinders; • non-cylindrical quadric surfaces.
17. **Component-wise formulas for vector functions:**
    - limits, derivatives, integrals; • tangent and unit tangent vector.
18. **Trigonometric identities.**
    (a) Half-angle formulas (deg. reduction): $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.
    (b) Double-angle formulas: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$.
    (c) Turning products into sums: $\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$;
    $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$; $\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$.
19. **Some formulas for arc-length and surface of revolution:**
    - If $f(x)$ is a function on $[a, b]$ such that $f'(x)$ is continuous on $[a, b]$, then the *arc length* of the curve $y = f(x)$ is $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$; and the *surface area* S of the solid obtained by revolving $y = f(x)$ about the x-axis is $S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$.
    - If $\vec{r}(t)$ is a vector function traced once by $t \in [a, b]$, then its arc length is $L = \int_a^b |\vec{r}\,'(t)| dt$.
20. **Differentiation Laws for vector functions:** DL±; PR·c; PR·$f(t)$; PR∘; PR×; CR.
21. **Unit tangent, normal, binormal vectors:** If $\vec{r}(t)$ is a *smooth* vector function, then
    - $\overrightarrow{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$; • $\overrightarrow{N}(t) = \frac{\overrightarrow{T}\,'(t)}{|\overrightarrow{T}\,'(t)|}$; • $\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$.
22. **Fundamental problems for vector functions:** (assuming all vectors below exist)
    - If $|\vec{r}(t)|$ is a constant, how are the tangent $\vec{r}\,'(t)$ and the position vector $\vec{r}(t)$ related?
    - How about the unit tangent vector $\overrightarrow{T}(t)$ and its derivative $\overrightarrow{T}\,'(t)$?
    - If $\vec{r}(t) \neq \vec{0}$ and $\vec{r}\,'(t)$ exists, what is $|\vec{r}\,(t)|'$?
    - Why is $|\overrightarrow{T}(t) \times \overrightarrow{T}\,'(t)| = |\overrightarrow{T}\,'(t)|$? Connection with $\overrightarrow{N}(t)$ and $\overrightarrow{B}(t)$?
23. **Relations with the arc-length function $s(t)$:**
    - $s'(t) = |\vec{r}\,'(t)|$ is the speed along the curve;
    - $\vec{r}\,'(t) = s'(t) \cdot \overrightarrow{T}(t)$; • $\vec{r}\,''(t) = s''(t) \cdot \overrightarrow{T}(t) + s'(t) \cdot \overrightarrow{T}\,'(t)$;
    - $\overrightarrow{T}(s) = \vec{r}\,'(s)$; • $L = \int_a^b |\overrightarrow{T}(s)|\, ds$; • $\kappa(s) = |\overrightarrow{T}\,'(s)| = |\vec{r}\,''(s)|$.
24. **Curvature:** The curvature of a vector function $\vec{r}(t)$ is given by $\kappa(t) = \frac{|\overrightarrow{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.
    In the special case of a plane curve $y = f(x)$, the curvature equals $\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$.
25. **Coordinate planes "in motion":** $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where (x_0, y_0, z_0) is in the plane, and $\langle a, b, c\rangle = \overrightarrow{T}, \overrightarrow{B}, \overrightarrow{N}$ for the normal, osculating, and rectifying planes.
26. **Acceleration:** $\vec{a}(t) = a_T\overrightarrow{T} + a_N\overrightarrow{N}$. Moreover, with speed $\nu = s'(t)$ and curvature κ:
    - $a_T = \nu' = \frac{\vec{r}\,'(t) \circ \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$; • $a_N = \kappa\nu^2 = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$.

3 Problem Solving Techniques

1. **Convert between coordinate systems**
   - $(x, y) \mapsto (\sqrt{x^2 + y^2}, \arctan \frac{y}{x})$ and $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$.
2. **Find Tangent Slopes/Lines to a Parametric Curve given by** $y = y(t)$ **and** $x = x(t)$**:**
   - If $\frac{y'(t)}{x'(t)}$ is well-defined (i.e., both top and bottom quantities exist, are finite numbers, and $x'(t) \neq 0$), this is the tangent slope.
   - If $x'(t) = 0$ but $y'(t) \neq 0$, we have a vertical tangent line.
   - If $x'(t) = 0 = y'(t)$, apply LH to $\frac{y'(t)}{x'(t)}$ and repeat the process for the resulting quotient.
3. **Change lengths of vectors by dot products:**
   - $|\vec{v}|^2 \mapsto \vec{v} \circ \vec{v}$; • $|\vec{v}| \mapsto \sqrt{\vec{v} \circ \vec{v}}$.
4. **Construct a Plane given by three points** P, Q, R **on it**
   1. Construct the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.
   2. Find the normal vector $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$.
   3. Let $\vec{n} = \langle a, b, c\rangle$ and $P = (x_0, y_0, z_0)$. Then the plane containing P, Q, R is given by
   $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \implies ax + by + cz = d,$$
   where we moved all the constants to one side to simplify the equation.
5. **Find Distances**
   - **Point to Point:** The distance from a point P to a point Q is found by taking the magnitude of the vector from P to Q; i.e., $|\overrightarrow{PQ}|$.
   - **Point to Line:** Let P be a point and L be a line.
     1. Choose a point Q on L and construct the vector $\overrightarrow{PQ}$.
     2. Find a vector $\vec{v}$ that is parallel to the line.
     3. The distance is $\frac{|\overrightarrow{PQ} \times \vec{v}|}{|\vec{v}|}$.
   - **Point to Plane:** Let P be a point and let $\mathcal{P}$ be a plane. Suppose the equation for the plane is $ax + by + cz + d = 0$ and $P = (x_0, y_0, z_0)$. Then the distance $\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.
   - **Line to Line:** Let L_1 and L_2 be the two lines.
     1. Check if the lines intersect. If so, then distance is 0; otherwise, move on to step 2.
     2. Let $\vec{v}_1$ be the vector parallel to L_1 and $\vec{v}_2$ be the vector parallel to L_2.
     3. Let $\vec{n} = \vec{v}_1 \times \vec{v}_2$. This vector is the normal vector to the plane containing L_1 and the plane containing L_2 (so these planes are parallel).
     4. Pick a point $Q = (x_2, y_2, z_2)$ on L_2 and use the vector $\vec{n}$ to write the equation for the plane $\mathcal{P}_2$ containing L_2: $ax + by + cz + d = 0$.
     5. Pick a point $P = (x_1, y_1, z_1)$ on L_1. Now apply the algorithm for point-to-plane using P and $\mathcal{P}_2$.
   - **Plane to Plane:** Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two planes.
     1. Choose a point P on $\mathcal{P}_1$.
     2. Apply the algorithm for point-to-plane using P and $\mathcal{P}_2$.
6. **Graph Quadric Surfaces**
   1. Given a general equation, complete the squares for all variables that have a squared term so that no mixed terms are left and at most one linear term is left.
   2. Consider the same surface that is "centered" at the origin by simplifying the equation to one of the 12 standard equestions, and name the surface.
   3. By setting variables to appropriate constants, describe the resulting traces.
   4. Note any values of x, y, or z that results in special traces; e.g., a point, a pair of lines, empty trace, etc.
   5. Shift back the surface to the reflect the original equation.

7. **Find Arc Length**
   1. Let $\vec{r}(t)$ be a vector function defined for $\alpha \leq t \leq \beta$. Differentiate $\vec{r}(t)$ to obtain $\vec{r}\,'(t)$.
   2. Evaluate $|\vec{r}\,'(t)|$.
   3. The arc length is $L = \int_\alpha^\beta |\vec{r}\,'(t)| dt$.
8. **Calculate Curvature**
   1. Let $\vec{r}(t)$ be the given vector function.
   2. Find $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.
   3. Then curvature is $\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$.
9. **Find Components of Acceleration**
   1. Let $\vec{r}(t)$ be the given vector function.
   2. Find $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.
   3. Then $\vec{a}(t) = \vec{r}\,''(t) = a_T \overrightarrow{T} + a_N \overrightarrow{N}$ where $a_T = \frac{\vec{r}\,'(t) \circ \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$ and $a_N = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|}$.

4 Problems for Review

The exam will be based on Homework, Lecture, Section and Quiz problems. Review **all** homework problems, and all your classnotes and discussion notes. Such a thorough review should be enough to do well on the exam. If you want to give yourself a mock-exam, select 4 representative problems from various HW assignments, give yourself 40 minutes, and then compare your solutions to the HW solutions. If you didn't manage to do some problems, analyze for yourself what went wrong, which areas, concepts and theorems you should study in more depth, and if you ran out of time, think about how to manage your time better during the upcoming exam.

4.1 Single-Variable Calculus in Other Coordinates

We looked at parametric representations of curves, polar coordinates, tangents to parametric and polar curves, and integral calculus (areas, arc lengths, surface areas of solids of revolution) on parametric and polar curves.

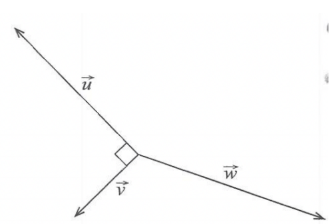
1. True/False practice:
   (a) The equations $x = r\cos\theta, y = 2r\sin\theta$ for some $r > 0$ represent an ellipse in polar coordinates.
   (b) The region $0 \leq \theta \leq \pi/4, 0 \leq r \leq 2$ is a circular sector.
   (c) In polar coordinates, instead of a vertical line test we have a "radial ray test;" a polar curve where you can connect the origin and two points on the curve with a straight line does not come from an expression of the form $r = f(\theta)$.
   (d) When finding the area between a parametric curve and the x-axis by setting up an integral, you should be careful with the bounds of your integral to ensure you get the right sign.
   (e) There is nothing mysterious at all about our formulas for the arc length of parametric and certain polar curves; they come from the usual cartesian formulas, our formulas for tangent slopes, and integration by substitution.
2. Sketch and describe in the words the regions in the plane defined by the following inequalities:
   (a) $2 \leq r \leq 4$;
   (b) $\pi \leq \theta \leq 2\pi$.
3. Find three distinct representations, one of which has $r < 0$, of the point with Cartesian coordinates $(1, \sqrt{3})$ in polar coordinates.

4. What is the slope of the tangent line to the curve $r = 4\theta^2$ at the point with the Cartesian coordinates $(0, \pi)$?
5. Find all points (x, y) in the plane where the curve $x = 2t^3 - 3t^2, y = t^4 - 4$ has horizontal or vertical tangents.
6. Use the formula for surface area of solids of revolution of parametric curves to prove that the surface area of a sphere of radius 1 is 4π.
7. Consider the curve C described in polar coordinates by $r = 2\sin\theta$.
   a) Sketch C in the xy-plane. Indicate the interval for θ over which the curve is traversed exactly once. Include all of your calculations.
   b) Find the area enclosed by C.
   c) Prove that C is a circle by finding its Cartesian equation.
8. Derive the formula for the surface area of a sphere or radius $r = 5$ by following the steps below. Include all calculations and relevant explanations.
   a) Sketch and parameterize a circle of radius $r = 5$ centered at the origin in the xy-plane.
   b) Now revolve the circle about the y-axis, and using your parameterization from part (a), find the surface area of the resulting solid.

4.2 Vectors and the Geometry of Space

We looked at the geometry of three-dimensional space $\mathbb{R}^3$, vectors and vector addition, the dot product, scalar and vector projection, direction cosines, the cross product, the determinant formula for cross products, the scalar triple product, equations for lines in $\mathbb{R}^3$ (vector, parametric, symmetric), equations for planes in $\mathbb{R}^3$, intersections of planes with lines and planes with planes, distances from points to lines and planes.

1. True/False practice:
   a) If you draw the positive x-axis going to the left along a sheet of paper and the positive y-axis going down towards the bottom of the sheet of paper, then to get a set of coordinate axes for $\mathbb{R}^3$, the positive z-axis would be coming straight up out of the paper.
   b) The expression $((\vec{a} \circ \vec{b})\vec{c}) \circ \vec{d}$ makes sense.
   c) The cross product is associative.
   d) It makes sense to talk about *the* normal direction to a line in $\mathbb{R}^3$.
   e) One way to think about the equation $ax + by + c = 0$ as a line in the plane is that it comes form knowing the vector $\langle a, b\rangle$ is orthogonal to the line and using some point (x_0, y_0) on the line.
2. Let $\vec{u} = \langle 1, -1, 3\rangle, \vec{v} = \langle 0, 2, -1\rangle, \vec{w} = \langle 2, 0, 1\rangle$. Evaluate $\vec{u} \circ (\vec{v} \times \vec{w})$.
3. Let the vectors in the figure satisfy $|\vec{u}| = 2$, $|\vec{v}| = 1$, and $\vec{u} + \vec{v} + \vec{w} = \vec{0}$.
   a) What is $|\vec{w}|$ equal to? Explain. (*Hint:* Express $\vec{w}$ in terms of the other vectors, and use this in other parts of the problem too!)



   b) What is the dot product $\vec{v} \circ \vec{w}$ equal to? Explain.
   c) What is the length of $\vec{v} \times \vec{w}$? Explain.
   (*Hint:* If α is the angle between the vectors, can you find $\cos\alpha$ from something above?)

4. Consider the line ℓ with symmetric equations given by $\frac{x-1}{2} = \frac{y+2}{3} = z$.
   a) Is it parallel, perpendicular, or skew to the line whose parametric equations are $x = t+4, y = 2-t, z = t-1$?
   b) What is the distance from the origin to the line ℓ?
   c) Write an equation for a plane perpendicular to this line.
5. In a trapezoid $ABCD$ it is known that base AB is 10 cm and the base DC is 6 cm. Let M be the midpoint of side AD and N the midpoint of side BC. The segment MN is called the *midsegment* of the trapezoid $ABCD$.
   a) Write $\overrightarrow{MN}$ as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{DC}$ and, using vectors, prove that this linear combination is correct.
   b) Prove that the midsegment MN is parallel to the bases of the trapezoid. Find its length. (Hint: You may use part (a).)
6. Consider the three points $P = (1, -2, 0), Q = (-1, 0, 3)$, and $R = (-3, 2, 0)$ in $\mathbb{R}^3$.
   a) What is $\cos \angle PQR$?
   b) Find an equation for the plane containing the three points P, Q, and R.
   c) What is the area of the triangle PQR?
7. Consider the two planes $x + 2y + 2z + 4 = 0$ and $3x - 4y + 12z = 0$.
   a) What is the cosine of the (acute) angle between the planes?
   b) Write symmetric equations of the line of intersection between the two planes.
8. Write an equation for the surface of revolution formed by rotating the curve $x = \sqrt{1+y^2}$ about the x-axis. What kind of surface is this?
9. One line L_1 passes through the point $P(1, 1, 0)$ and is parallel to the vector $\vec{v} = \langle 1, -1, 2\rangle$. A second line L_2 passes through the points $Q(0, 2, 2)$ and $R(1, 1, 2)$.
   a) Find the intersection point of the two lines.
   b) Find and equation of the plane $\mathcal{P}$ that contains these two lines.
   c) Find the distance from the origin to the plane $\mathcal{P}$.
10. Consider the surface $x = y^2 + 4y + z^2 + 5$. Sketch the surface and indicate your reasoning. Describe the cross-sections (traces) of the surface with planes perpendicular to any of the three axes, and make sure that these features are reflected in your sketch. Are there any special traces? What is the name of this surface? Include all relevant calculations and explanations.
11. Consider the surface $x^2 - 4y^2 + z^2 + 4x - 8y - 6z = -13$. Sketch the surface and indicate your reasoning. Describe the cross-sections (traces) of the surface with planes perpendicular to any of the three axes, and make sure that these features are reflected in your sketch. Are there any special traces? What is the name of this surface? Include all relevant calculations and explanations.

4.3 Calculus with Vector-Valued Functions

We looked at vector functions, limits of vector functions, derivatives of vector functions, unit tangent vectors, differentiation laws (sum/difference rule, chain rule, product rules) for vector functions, integrals of vector-valued functions, arc length of vector-valued functions.

1. True/False practice:
   a) The domain of the function $\vec{u}(t) = \vec{v}(t) \circ \vec{w}(t)$ is $[0, 2]$ if the domain of $\vec{v}(t)$ is $t \geq 0$ and domain of $\vec{w}(t)$ is $t \leq 2$.
   b) The sum of two differentiable vector functions is differentiable.
   c) To find the definite integral from $t = a$ to $t = b$ of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, we take the vector with components the definite integral $\int_a^b f(t)dt$, $\int_a^b g(t)dt$, and $\int_a^b h(t)dt$.

d) The arc length between two points in $\mathbb{R}^3$ of a curve given by a vector function depends on the parameterization of the curve; if we replace t with $2t$, we'll double the arc length since the new curve is going twice as fast.

e) When we renormalize a tangent vector to find a unit tangent vector to a curve at a given point, it is okay if we multiply by -1 since multiplying a unit vector -1 gives a unit vector so multiplying by -1 doesn't change anything.

2. Two flies move through space with flight paths described by the vector functions $\vec{r}_1(t) = \langle 2t+1, 3+\cos\pi t, 4t\rangle$ and $\vec{r}_2(t) = \langle t, 3-\sin\pi t, t^2-2\rangle$.
   a) At the time $t = 3$, which fly is flying faster?
   b) Do their paths intersect?
   c) Do the flies collide?
3. Find symmetric equations for the tangent line to $\vec{r}(t) = \langle 3t^2, 1+e^{t-1}, t^{-1}\rangle$ at point $(3,2,1)$.
4. Consider three vector-valued functions $\vec{a}(t), \vec{b}(t), \vec{c}(t)$ taking values in $\mathbb{R}^3$. What is the derivative of the triple product $\vec{a}(t) \circ (\vec{b}(t) \times \vec{c}(t))$?
5. A particle moves with position vector described by $\vec{r}(t) = \langle \cos t, \sin t, \ln\sec t\rangle$ for $0 \le t \le \pi/4$. How far does the particle travel between $t = 0$ and $t = \pi/4$.
6. A spaceship is traveling along the twisted cubic C with position function $\vec{r}(t) = \langle t, t^2, t^3\rangle$ for all $t \in \mathbb{R}$. Below, include all relevant calculations and explanations.
   a) Find the tangential component of the spaceship's acceleration along C at the origin.
   b) Find the normal component of the spaceship's acceleration along C at the origin.
   c) Write the spaceship's acceleration vector at the origin in terms of the local coordinate unit vectors $\overrightarrow{T}$, $\overrightarrow{N}$, and $\overrightarrow{B}$.
7. Prove, using the methods we learned in this course, that the tangent line at a point A to a circle with center O is perpendicular to the radius OA.

4.4 Extra True/False Practice

1. True/False practice:
   a) The vector function $\vec{r}(t) = \langle t, t^2\rangle$ contains more information than the Cartesian function $y = x^2$.
   b) Any three points in $\mathbb{R}^3$ determine a unique plane.
   c) Any time $dx/dt = 0$ for a parametric curve, the curve has a vertical tangent.
   d) If $\vec{u}(t)$ and $\vec{v}(t)$ are vector functions, then $\vec{u}(t) \times \vec{v}(t)$ is a vector function.
   e) Any line in $\mathbb{R}^3$ will intersect one of the three coordinate planes $x = 0, y = 0, z = 0$.
   f) Let P_1, P_2, and P_3 be three distinct planes in $\mathbb{R}^3$. Then the intersection of all three planes, $P_1 \cap P_2 \cap P_3$ is either the empty set, a single point, or a line.
   g) We say that the graph of $xz = 1$ in $\mathbb{R}^3$ is a cylinder even though the traces for constant y are hyperbolas and not circles.
   h) The arc length formula for a vector function $\vec{r}(t)$ in $\mathbb{R}^3$, $L = \int_\alpha^\beta |\vec{r}(t)|dt$, is true for the same reasons as our arc length formulas for cartesian, polar, and parametric curves even though, when written this way, there is no square root sign.
   i) We can parametrize a hyperbola much the same way as we parametrize an ellipse, replacing cos with cosh and sin with sinh.
   j) if the lines with vector equations $\vec{r}_0 + t\vec{v}$ and $\vec{r}_1 + s\vec{w}$ are skew, there is *nothing* we can say about $\vec{v} \times \vec{w}$.
   k) The centroid of a triangle can be found by intersecting just *two* of the triangle's medians.
   l) If a parametric curve given by $x = f(t)$, $y = g(t)$ satisfies $g'(0) = 0$, then the curve has a horizontal tangent at $t = 0$.

m) The curve with vector equation $\vec{r}(t) = 6\,t^5\,\vec{i} - t^5\vec{j} + 3\,t^5\vec{k}$ is a line.
n) Among quadric surfaces, there are *three* types of "elliptic hyperboloids," but none of them is called by this name.
o) The polar curves $r = 1 - \cos(2\theta)$ and $r = \cos(2\theta) - 1$ have the same graph.
p) For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ we have $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$.
q) If $\vec{v}$ and $\vec{w}$ are vectors in the plane, then the equality $|\vec{v} + \vec{w}| = |\vec{v}| + |\vec{w}|$ is satisfied only when $\vec{v}$ and $\vec{w}$ are in a special position relative to each other.
r) There are at least *three* "product rules" for differentiation that involve vector functions.
s) If $|\vec{r}(t)| = 7$ for all t, then $\vec{r}\,'(t) \perp \vec{r}(t)$ for all t.
t) Different parameterizations of the same smooth curve result in identical *unit tangent* vectors on the curve.
u) Given a curve $x = f(t)$ and $y = g(t)$, with both $f'(t)$ and $g'(t)$ continuous and both $f(t) \geq 0$ and $g(t) \geq 0$ for $a \leq t \leq b$, the two formulas for the *surface areas* of the solids of revolution about the x-axis and the y-axis differ only in one place.
v) It is straightforward to convert from *Cartesian* to *polar* coordinates; but we have to pay attention to the quadrant where the point is when converting from *polar* to *Cartesian* coordinates because we could end up with the wrong angle θ.
w) A shortcut formula for the *tangential* component of the acceleration vector can be turned into a shortcut formula for the *normal* component by changing one vector multiplication operation to another in the numerator and adding absolute values there to ensure that we will obtain overall a number.
x) The equation defining any *cylinder* in $\mathbb{R}^3$ is necessarily missing one of the variables x, y, or z that corresponds to the ruling of the cylinder.
y) *Exactly one* of the following two statements is true:
(a) $|\vec{v} \circ \vec{w}| = |\vec{v}| \cdot |\vec{w}|$ for *some* vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$;
(b) $|\vec{v} - \vec{w}| = |\vec{v}| - |\vec{w}|$ for *all* vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$.

5 No Calculators during the Exam. Cheat Sheet and Studying for the Exam

No Calculators will be allowed during the exam. Anyone caught using a calculator will be disqualified from the exam.

For the exam, you are allowed to have a "cheat sheet" - *one page* of a regular 8.5×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.) DSP students with special writing or related disability should consult with the instructor regarding their cheat sheets.
- You must submit your cheat sheet on Gradescope by 11AM on the day of the exam.
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the exam.
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! No sleeping the night before the exam due to cramming, or more than 3 hours of math study on the day of the exam is counterproductive! No kidding!**